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MODAL EQUATIONS FOR THE NONLINEAR FLEXURAL VIBRATIONS OF A CYLINDRICAL SHELL

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Abstract—A review is made of modal approximations in deriving the equations of motion for the nonlinear flexural vibrations of a cylindrical shell. An improved method is proposed and used which satisfies more exactly the boundary conditions of the problem. Comparisons are made between the present equations and those previously derived in the literature.

NOTATION

a	plate length
b	plate width
$A_{M0}, A_{MN}, B_{MN}, A_{11}, B_{21}, C_{21}$	modal amplitudes
$a_{M0}, a_{MN}, a_{MN}, a_{11}, b_{21}, c_{21}$	nondimensional modal amplitudes
D	plate or shell stiffness
E	modulus of elasticity
h	plate or shell thickness
L	length of shell
M	moment
N_x, N_y, N_{xy}	stress resultants
M, N, m, n	mode numbers
R	radius of cylinder
t	time
u, v, w	plate or shell displacements
x, y	coordinates of plate or shell
Γ	$\equiv L^2/hR$
γ	$= \dfrac{\xi^4\left[\dfrac{1}{(\xi^2+1)^2}-\dfrac{1}{16}-\dfrac{\varepsilon}{12(1-\nu^2)}\right]}{\left[\dfrac{\xi^4}{(\xi^2+1)^2}+\dfrac{\varepsilon(\xi^2+1)^2}{12(1-\nu^2)}\right]}$
δ	$= \dfrac{3\xi^4}{16}\dfrac{\left[\dfrac{1}{(\xi^2+1)^2}+\dfrac{1}{(9\xi^2+1)^2}\right]}{\left[\dfrac{\xi^4}{(\xi^2+1)^2}+\varepsilon\dfrac{(\xi^2+1)^2}{12(1-\nu^2)}\right]}$
η	$= 3(1-\nu^2)$

ε	$= (N^2h/R)^2$
Φ	stress function
ν	Poisson's ratio
ω	frequency
ω_N^2	$= DN^4/\rho_m hR^4$
ω_{MN}^2	$= \dfrac{E}{\rho_m R^2}\left[\dfrac{\xi^2}{(\xi^2+1)^2} + \varepsilon\dfrac{(\xi^2+1)^2}{12(1-\nu^2)}\right]$
Ω	$= \omega/\omega_N$
ξ	$= \dfrac{M\pi/L}{N/R}$
ρ_m	material density
τ	$= \left[\dfrac{D}{\rho_m ha^4}\right]^{\frac{1}{2}} t$ or $\left[\dfrac{D}{\rho_m hL^4}\right] t$
τ_{MN}	$= \omega_{MN} t$
τ_N	$= \omega_N t$

1. INTRODUCTION

THE nonlinear flexural vibrations of plates and shells have recently received renewed attention in the literature. This is probably because of two reasons:

(1) the growing appreciation of the importance of nonlinear effects in determining the stability and response of thin shells (including plates) under dynamic loading, and
(2) our increased ability to handle a more complex description of such structural members via high speed computers.

Nevertheless before one can apply (2) to investigate the many interesting ramifications of (1), it is necessary to obtain a proper and consistent formulation of the modal equations of motion. This task (using Galerkin's method, for example) turns out to be a fairly subtle and moderately difficult one for some shell geometries. In particular, the cylindrical shell is most challenging as has been clearly pointed out by Evensen [1, 2]. The primary difficulty lies in satisfying certain continuity and boundary conditions for the shell.

The purpose of the present paper is to satisfy more accurately these boundary and continuity conditions and to investigate their effect on the form of the modal equations. Galerkin's method is employed. In Section 2 we review the results for a flat plate and also a ring which may be considered limiting cases for a cylindrical shell. In Section 3 the cylindrical shell per se is treated and in Section 4 the present results are compared with those of previous authors.

2. RELATED RESULTS FOR PLATES AND RINGS

2.1 *Nonlinear vibrations of plates*

To appreciate the difficulty with the cylindrical shell, it is perhaps best to begin a discussion with the work on the flat plate. (Some of the plate results discussed here are thought to be original.) The paper by Chu and Herrmann [3] is one of the earliest and in many ways representative of the subsequent literature. Chu and Herrmann arrive at the equations of motion by first doing a perturbation analysis to show the consistency of neglecting in-plane intertia terms in the study of the nonlinear flexural vibrations of a plate. This portion of their work is analogous to that of Reissner [4] who considered the linear flexural vibrations of a curved shallow shell. With this simplification, the equations of motion reduce to the well-known Von Karman equations including only the transverse inertia term. Chu and Herrmann then proceed to solve for the fundamental frequency of vibration as a function of plate amplitude using a one term modal solution via a Rayleigh–Ritz or Galerkin type procedure. The boundary conditions which they satisfied were

$$w = 0 \quad \text{on all edges} \tag{2.1}$$

$$M = 0 \quad \text{on all edges} \tag{2.2}$$

$$u = 0 \quad \text{on both edges normal to } u \tag{2.3}$$

$$v = 0 \quad \text{on both edges normal to } v \tag{2.4}$$

$$N_{xy} = 0 \quad \text{on all edges.} \tag{2.5}$$

The last boundary condition was not explicitly stated in [3]; however, it may be verified that their solution does satisfy this condition. Of course, unless four boundary conditions are specified on each of the four plate edges, the solution is not unique.

Subsequent investigators, while using a modal approach, have handled the boundary conditions in a different way. Bolotin [5], Fralich [6], and Dowell [7], who have studied the problem *inter alia*, have satisfied boundary conditions (2.1) and (2.2) exactly but satisfied (2.3) to (2.4) only "on the average" by requiring the integral of the appropriate quantity, u, v, or N_{xy} to be zero along the relevant edge. The solutions have been worked out for two [5, 6] and an arbitrary number of modes [7] respectively. The interesting result is that when only one mode is used in each of these analyses [5–7], the resulting equation of motion is *identical* to that of Chu and Herrmann. In all cases, the equation of motion is

$$\frac{d^2 a_{11}}{d\tau^2} + \pi^4[1+(a/b)^2]a_{11} + 3a_{11}^3\pi^4\left\{\left(\frac{3}{4}-\frac{v^2}{4}\right)[1+(a/b)^4]+v(a/b)^2\right\} = 0 \tag{2.6}$$

where $w = A_{11}\sin \pi x/a \sin \pi y/b$ and $a_{11} = A_{11}/h$.

The boundary conditions cited above are not the only possible ones of interest, of course. Therefore others have been considered. For example, it may be more realistic to consider in place of (2.3) to (2.5)

$$u = 0 \quad \text{on all edges} \tag{2.3a}$$

$$v = 0 \quad \text{on all edges.} \tag{2.4a}$$

If these latter two conditions are satisfied "on the average", one finds that for a one mode analysis the result is still equation (2.6). If, on the other hand, (2.3a) and (2.4a) are satisfied exactly by taking the following expansions

$$\begin{aligned} u &= B_{21}\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b}, \\ v &= C_{12}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b}. \end{aligned} \tag{2.7}^*$$

* $B_{11} = C_{11} = 0$, as is perhaps obvious from symmetry.

The equations of motion become, using Galerkin's method,

$$\frac{d^2a_{11}}{d\tau^2}+\pi^4[1+(a/b)^2]^2a_{11}+3a_{11}^3\pi^4\left\{\frac{3}{8}[1+(a/b)^4]+\left(\frac{\nu-1}{4}\right)(a/b)^2\right\}$$

$$-\pi_2^2b_{21}a_{11}\left\{\frac{2}{3}[1+\nu(a/b)^2]+\frac{1-\nu}{6}(a/b)^2\right\}$$

$$+\pi^2c_{21}a_{11}\left\{\frac{2}{3}[(a/b)^2+\nu]+\frac{(1-\nu)}{6}\right\}=0 \tag{2.8}$$

where $b_{21}\equiv(B_{21}/h)a/h$, $c_{12}\equiv(C_{12}/h)a/h$ are determined from

$$b_{21}\left\{1+\frac{(1-\nu)}{8}(a/b)^2\right\}+(1+\nu)\frac{(4/3)^2}{\pi^2}(a/b)c_{12}=a_{11}^2\left\{-\tfrac{1}{3}+(a/b)^2\left(\frac{3\nu-1}{12}\right)\right\}, \tag{2.9}$$

$$c_{12}\left\{(a/b)^2+\frac{(1-\nu)}{8}\right\}+(1+\nu)\frac{(4/3)^2}{\pi^2}(a/b)b_{21}=a_{11}^2(a/b)\left\{-\tfrac{1}{3}(a/b)^2+\left(\frac{3\nu-1}{12}\right)\right\}. \tag{2.10}$$

It is a simple matter to solve for b_{21} and c_{12} from equations (2.9) and (2.10) and to substitute into (2.8) to obtain an equation solely in terms of a_{11}. However it is apparent that the resulting equation will be too complicated to permit any simple analytical comparison with equation (2.6). Instead numerical comparisons have been made for the coefficient of the nonlinear terms in equations (2.6) and (2.8) for $a/b=0$ and 1·0. In both cases, the results are in very close agreement with the coefficient from equation (2.8), being slightly larger than from (2.6).

Finally to further verify the generality of the above conclusions, a clamped plate was considered where boundary condition (2.2) was replaced by

$$\frac{\partial w}{\partial x}=0 \qquad \text{for } x=0,a,$$

$$\frac{\partial w}{\partial y}=0 \qquad \text{for } y=0,b. \tag{2.2a}$$

Again there was similar agreement among the various results using different in-plane boundary conditions.

Hence one concludes that

(i) the important in-plane boundary conditions are (2.3) and (2.4). (2.5), apparently, is not very important.

(ii) satisfying boundary conditions (2.3), (2.4), and (2.5) "on the average" is a good approximation.

It should be mentioned, however, with regard to (ii) that comparisons have also been made between "exact" and "average" boundary condition solutions for *zero stress on the edges* which show poor agreement. The nonlinear coefficient predicted by the "average" solution is much higher than the "exact" boundary conditions solution. For this problem, boundary conditions (2.3) and (2.4) were replaced by

$$N_x=0, \tag{2.3b}$$

$$N_y=0. \tag{2.4b}$$

In order to satisfy these conditions, along with (2.5), exactly the following stress function was chosen:

$$\Phi = D\left[1-\cos\frac{2\pi x}{a}\right]\left[1-\cos\frac{2\pi y}{b}\right], \tag{2.11}$$

where D is a constant to be determined. Again, using Galerkin's method, one obtains

$$\frac{d^2a_{11}}{d\tau^2}+\pi^4[1+(a/b)^2]^2a_{11}+3a_{11}^3\pi^4\left\{\frac{(1-\nu^2)(a/b)^4}{[1+(a/b)^2]^2+2[1+(a/b)^4]}\right\} = 0. \tag{2.12}$$

If one satisfies the boundary conditions "on the average", one obtains

$$\frac{d^2a_{11}}{d\tau^2}+\pi^4[1+(a/b)^2]^2a_{11}+3a_{11}^3\pi^4\left\{\frac{(1-\nu^2)}{4}[1+(a/b)^4]\right\} = 0. \tag{2.13}$$

It will be noted that, in equation (2.12), the nonlinear coefficient vanishes when $a/b \to 0$ as one would expect from the well-known results for a beam with no in-plane restraint. On the other hand, in equation (2.13), the nonlinear coefficient remains finite as $a/b \to 0$. To make a further comparison, we give in Table 1 the numerical values of the nonlinear coefficient, $\{\ldots\}$, as determined from equations (2.12), (2.13) and, for reference, (2.6) for two values of a/b, 0 and 1·0, and $\nu = 0{\cdot}3$.

TABLE 1

a/b	(2.6)	(2.12)	(2.13)	Zero stress	Zero strain
0	0·728	0·0	0·228	0·91	1·0
1·0	1·755	0·114	0·450	—	—

While both equations (2.12) and (2.13) indicate a much decreased nonlinearity from equation (2.6), the quantitative agreement between the former two results is poor. Equation (2.12) is undoubtedly the more accurate result, giving as it does the correct limit for $a/b \to 0$.

One final comment concerning the limit $a/b \to 0$ is in order. As is known, if one considers no spanwise bending, $\partial w/\partial y = 0$, and takes either zero spanwise stress or strain but $u = 0$ at $x = 0$, a, an equation similar to equation (2.6) may be derived (see, for example, [7]). For the sake of completeness, the nonlinear coefficients of these results are also given in Table 1. Note that neither agrees precisely with equation (2.6), the difference being roughly 25%.

2.2 *Nonlinear vibrations of rings*

Now, for the plate, we are able to obtain reasonably satisfactory results for the nonlinear oscillations in the fundamental mode by retaining only one term in the series for w. However Evensen [8] has shown (and Dowell [9] has also confirmed) that for the ring it is essential to consider the coupling between the axisymmetric circumferential mode and the particular circumferential mode being studied, say $n = N$. The first author has discussed this problem elsewhere [9], and we will only briefly recapitulate the relevant results here. The equations for a ring are [8, 9]

$$D\frac{\partial^4 w}{\partial y^4}+\rho_m h\frac{\partial^2 w}{\partial t^2} = \frac{N_y}{R}+N_y\frac{\partial^2 w}{\partial y^2}, \tag{2.14}$$

$$\frac{\partial N_y}{\partial y} = 0 \tag{2.15}$$

where

$$N_y = Eh\left[\frac{\partial v}{\partial y} - \frac{w}{R}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]. \tag{2.16}$$

Actually Evensen [8] and Dowell [9] have considered a modified version of equation (2.14), viz.,

$$D\left(\frac{\partial^2}{\partial y^2}+\frac{1}{R^2}\right)\left(\frac{\partial^2 w}{\partial y^2}+\frac{w}{R^2}\right)+\rho_m h\frac{\partial^2 w}{\partial t^2}-\frac{N_y}{R}-N_y\frac{\partial^2 w}{\partial y^2} = 0 \tag{2.14a}$$

after a suggestion of Morley's [10]. This modification is not essential to what follows and, for consistency with the subsequent cylindrical shell analysis, will not be used here.

If we take

$$w = A_N \cos\frac{N_y}{R}+B_N \sin\frac{N_y}{R}+A_0,$$

then, noting N_y is only a function of time from equation (2.15), we may solve for N_y from equation (2.16) requiring at the same time that

$$v(2\pi R, t)-v(0, t) = \int_0^{2\pi R}\frac{\partial v}{\partial y}\,\mathrm{d}y \equiv 0, \tag{2.17}$$

i.e., v must be continuous. The result is

$$N_y = Eh\{\tfrac{1}{4}(N/R)^2[A_N^2+B_N^2]-A_0/R\}. \tag{2.18}$$

With this result and using Galerkin's method, the equations of motion may be determined from equation (2.14) as

$$a_N+\eta a_N[a_N^2+b_N^2]-a_0 a_N\frac{4\eta}{\varepsilon^{\frac{1}{2}}}+\frac{\mathrm{d}^2 a_N}{\mathrm{d}\tau_N^2} = 0,$$

$$b_N+\eta b_N[a_N^2+b_N^2]-a_0 b_N\frac{4\eta}{\varepsilon^{\frac{1}{2}}}+\frac{\mathrm{d}^2 b_N}{\mathrm{d}\tau_N^2} = 0,$$

$$-\frac{\eta}{\varepsilon^{\frac{1}{2}}}[a_N^2+b_N^2]+\frac{4\eta a_0}{\varepsilon}+\frac{\mathrm{d}^2 a_0}{\mathrm{d}\tau_N^2} = 0, \tag{2.19}$$

where

$$\eta \equiv 3(1-v^2) \qquad \varepsilon \equiv (N^2 h/R)^2$$

$$a_N \equiv A_N/h \qquad b_N \equiv B_N/h$$

$$a_0 \equiv A_0/h \qquad \tau_N \equiv \omega_N t$$

$$\omega_N \equiv \left[\frac{DN^4}{\rho_m h R^4}\right]^{1/2}.$$

If a_0 is neglected, it is apparent from equation (2.19a, b) that the nonlinearity is of the "hard spring" type since η is always positive. However, if a_0 is retained and we take

$$a_N = \bar{a}_N \cos \Omega\tau,$$

$$b_N = \bar{b}_N \sin \Omega\tau,$$

$$\Omega \equiv \omega/\omega_N,$$

and consider the region near the resonance point, $\Omega \approx 1$, we can solve for a_0 from equation (2.19c) and substitute the result in equations (2.19a) and (2.19b). These latter equations become (retaining lowest harmonics in Ω and lowest order terms in ε)

$$\bar{a}_N - \Omega^2 \bar{b}_N + \frac{\bar{a}_N}{4} \varepsilon \Omega^2 [-\bar{a}_N^2 + \bar{b}_N^2] = 0,$$

$$\bar{b}_N - \Omega^2 \bar{a}_N + \frac{\bar{b}_N}{4} \varepsilon \Omega^2 [-\bar{b}_N^2 + \bar{a}_N^2] = 0. \tag{2.20}$$

From equation (2.20), it is apparent the nonlinearity is of a softening type with a strength determined by the positive parameter, ε. Hence the retention of a_0 completely changes the character of the nonlinearity.* From this it is apparent that any analysis of the cylindrical shell which is valid for the long wavelength limit must include a_0.

Anticipating some later discussion, it is pointed out here that there is some difference in philosophy (though little in practice) between the solutions of Evensen [8, 11] and Dowell [9] for the ring. Evensen attributes the failure of earlier analyses to predict the correct type of nonlinearity to the neglect of the continuity condition on v. However, as indicated above, it is possible to satisfy the continuity condition and still obtain an incorrect result if a_0 is neglected. In Evensen's work, he took $N_y = 0$ (inextensionality assumption) for the ring. Both he and Dowell have verified that this is an excellent assumption for the ring. Under this assumption, as Evensen points out, the v continuity condition can only be satisfied if one retains a_0, in which case a_0 is determined in terms of a_N and b_N by requiring that v be continuous.

For future discussion, we record this latter result of Evensen. When $N_y = 0$, from equation (2.18)

$$A_0 = \frac{N^2}{4R}[A_N^2 + B_N^2], \tag{2.21}$$

and hence

$$w(y, t) = A_N \cos \frac{Ny}{R} + B_N \sin \frac{Ny}{R} + \frac{N^2}{4R}[A_N^2 + B_N^2]. \tag{2.22}$$

3. MODAL EQUATIONS FOR A CYLINDRICAL SHELL

3.1 *Method of analysis*

With the plate and ring results in mind, we shall now consider the cylindrical shell.

* Evensen [11] has shown that the retention of additional circumferential modes has little effect.

Donnell's shallow shell theory will be employed (see [12] for a derivation). w is positive inward.

$$D\nabla^4 w + \rho_m h \frac{\partial^2 w}{\partial t^2} = \frac{1}{R}\frac{\partial^2 \Phi}{\partial x^2} + \left(\frac{\partial^2 \Phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 \Phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 \Phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right), \tag{3.1}$$

$$\frac{\nabla^4 \Phi}{Eh} = -\frac{1}{R}\frac{\partial^2 w}{\partial x^2} + \left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]. \tag{3.2}$$

The results discussed in the previous section were obtained using the same equations with appropriate terms omitted.

For w we shall use the expansion

$$w(x, y, t) = \sum_m \sum_n A_{mn}(t) \cos\frac{ny}{R} \sin\frac{m\pi x}{L} + \sum_m \sum_n B_{mn}(t) \sin\frac{ny}{R} \sin\frac{m\pi x}{L}.$$

This expansion satisfies the simply-supported boundary conditions

$$w = 0 \quad \text{at } x = 0, L, \tag{3.3}$$

$$\frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } x = 0, L. \tag{3.4}$$

Obviously w is continuous in the y variable as well. The additional boundary and continuity conditions will be satisfied "on the average" in the spirit of [5, 6] and [7], since this has been shown to be a satisfactory approximation for a plate (or ring).

As a check on our results, we should retrieve (at least approximately) the results for a plate of $a/b = 0$ as the ratio of axial to circumferential wavelength becomes small, i.e.,

$$\frac{L/m}{\pi R/n} \to 0.$$

The other possible limit

$$\frac{L/m}{\pi R/n} \to \infty$$

should give us (at least approximately) the results derived by Evensen [8] (and subsequently by one of the authors [9]) for a ring. For both limits, it will be shown that the averaging procedure provides results in reasonable agreement with the known results.

From the general expansion, we shall retain the following terms:

$$w(x, y, t) = A_{MN} \cos\frac{Ny}{R} \sin\frac{M\pi x}{L} + B_{MN} \sin\frac{Ny}{R} \sin\frac{M\pi x}{L} + A_{M0} \sin\frac{M\pi x}{L}.$$

(For some applications, such as a flutter analysis, additional axial, though probably not circumferential modes, need to be retained.) Substituting the above into the right-hand side of equation (3.2), we may solve for Φ as

$$\Phi = \Phi_{\text{homogeneous}} + \Phi_{\text{particular}}$$

where

$$\frac{\Phi_{\text{particular}}}{Eh} = \left(\frac{M\pi}{L}\right)^2 \frac{1}{R}\left\{\left[\left(\frac{N}{R}\right)^2+\left(\frac{M\pi}{L}\right)^2\right]\left[A_{MN}\cos\frac{Ny}{R}\sin\frac{M\pi x}{L}+B_{MN}\sin\frac{Ny}{R}\sin\frac{M\pi x}{L}\right]\right.$$
$$+\left(\frac{M\pi}{L}\right)^4 A_{MO}\sin\frac{M\pi x}{L}\bigg\}+\left(\frac{M\pi}{L}\right)^2\left(\frac{N}{R}\right)^2\left\{\frac{A_{MN}^2}{2}\left[-\left(\frac{2N}{R}\right)^{-4}\cos\frac{2Ny}{R}\right.\right.$$
$$\left.+\left(\frac{2M\pi}{L}\right)^{-4}\cos\frac{2M\pi x}{L}\right]-A_{MN}B_{MN}\left(\frac{2N}{R}\right)^{-4}\sin\frac{2Ny}{R}+\frac{B_{MN}^2}{2}$$
$$\left.\times\left[\left(\frac{2N}{R}\right)^{-4}\cos\frac{2Ny}{R}+\left(\frac{2M\pi}{L}\right)^{-4}\cos\frac{2M\pi x}{L}\right]\right\}. \tag{3.5}$$

For the homogeneous solution, we will take

$$\Phi_{\text{homogeneous}} = \tfrac{1}{2}\bar{N}_x + \tfrac{1}{2}\bar{N}_y x^2 - \bar{N}_{xy}xy. \tag{3.6}$$

This is not the most general homogeneous solution (though it may be considered the lowest order terms of a power series expansion of the general solution), however, it will be sufficient to satisfy the relevant in-plane continuity and boundary conditions "on the average". The advantage of the averaging process is the simplicity of equation (3.6). The stress–displacement relations are

$$(1-\nu^2)\frac{N_x}{Eh} = -\frac{\nu w}{R}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y},$$
$$(1-\nu^2)\frac{N_y}{Eh} = -\frac{w}{R}+\frac{\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x},$$
$$(1-\nu^2)\frac{N_{xy}}{Eh} = 2(1-\nu)\left[\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]. \tag{3.7}$$

For our boundary and continuity conditions, we shall require that

$$\int_0^{2\pi R}\int_0^L \frac{\partial u}{\partial x}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi R}[u(L,y)-u(0,y)]\,\mathrm{d}y = 0,$$
$$\int_0^L\int_0^{2\pi R}\frac{\partial v}{\partial y}\,\mathrm{d}y\,\mathrm{d}x = \int_0^L [v(x,2\pi R)-v(x,0)]\,\mathrm{d}x = 0,$$
$$\int_0^{2\pi R}\int_0^L N_{xy}\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{3.8}$$

The first of these states that the axial displacements "on the average" are zero at $x = 0, L$; the second that the v displacement is continuous in the circumferential coordinate "on the average"; and the last that the average shear is zero. An alternative interpretation of (3.8 c) since

$$\int_0^L\int_0^{2\pi R}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = 0^*,$$

* w is periodic in y, hence the integral over y will be zero. This may be verified explicitly for the expansions of w used in the present paper.

is that

$$\int_0^L [u(2\pi R, x)-u(0, x)]\,dx+\int_0^{2\pi R} [v(L, y)-v(0, y)]\,dy = 0. \tag{3.9}$$

The first term (if the second is omitted) states that u is continuous in the circumferential variable while the second (if the first is omitted) states that the circumferential displacement is zero at $x = 0, L$ "on the average". Thus, if both of these conditions are satisfied together, it is equivalent to equation (3.8c).

Applying the boundary conditions, equation (3.8), and using equation (3.5) through (3.7), we may determine the unknown constants, $\bar{N}_x$, $\bar{N}_y$, $\bar{N}_{xy}$ as

$$(1-\nu^2)\frac{\bar{N}_x}{Eh} = \frac{[A_{MN}^2+B_{MN}^2]}{L^2}\left[M^2\frac{\pi^2}{8}+\nu N^2\left(\frac{L}{R}\right)^2\frac{1}{8}\right] - \nu\frac{A_{M0}}{L}[1-(-1)^M]\frac{L}{R}\frac{1}{M\pi}+2\frac{A_{M0}^2}{L^2}M^2\frac{\pi^2}{8},$$

$$(1-\nu^2)\frac{\bar{N}_y}{Eh} = \frac{[A_{MN}^2+B_{MN}^2]}{L^2}\left[N^2\left(\frac{L}{R}\right)^2\frac{1}{8}+\nu M^2\frac{\pi^2}{8}\right] - \frac{A_{M0}}{L}\frac{L}{R}[1-(-1)^M]\frac{1}{M\pi}(1-\nu^2) - \frac{A_{M0}}{L}\frac{L}{R}[1-(-1)^M]\frac{1}{M\pi}+\nu^2 2\frac{A_{M0}^2}{L^2}M^2\frac{\pi^2}{8},$$

$$\bar{N}_{xy} \equiv 0. \tag{3.10}$$

With Φ completely determined in terms of A_{MN}, B_{MN} and A_{M0}, we shall determine the equations of motion from equation (3.1) via Galerkin's method. Using the expansion for w and equation (3.10), equation (3.1) is weighted in turn by

$$\cos\frac{Ny}{R}\sin\frac{M\pi x}{L}, \qquad \sin\frac{Ny}{R}\sin\frac{M\pi x}{L}, \qquad \sin\frac{M\pi x}{L}$$

and integrated over the shell surface. The results are, in non-dimensionalized notation,

$$\left[(M\pi)^2+N^2\left(\frac{L}{R}\right)^2\right]^2 a_{MN}+\frac{d^2a_{MN}}{d\tau^2}+\frac{(M\pi)^4\Gamma^2 4\eta a_{MN}}{[(M\pi)^2+N^2(L/R)^2]^2} + \left(\frac{\bar{N}_x L^2}{D}\right)(M\pi)^2 a_{MN}+\left(\frac{\bar{N}_y L^2}{D}\right)N^2\left(\frac{L}{R}\right)^2 a_{MN} + 4\eta a_{MN}\frac{[a_{MN}^2+b_{MN}^2]}{16}\left[(M\pi)^4+N^4\left(\frac{L^4}{R}\right)\right] - \frac{4\eta 4}{3}\frac{[1-(-1)^M]}{M\pi}\left(\frac{L}{R}\right)^2 N^2\Gamma a_{M0}a_{MN} \times\left\{1+\frac{(M\pi)^4}{[(M\pi)^2+N^2(L/R)^2]^2}+\frac{(M\pi)^4}{[(2M\pi)^2+N^2(L/R)^2]^2}\right\} + 4\eta(M\pi)^4 a_{M0}^2 a_{MN}\left\{\frac{1}{2}+\frac{1}{4}\frac{N^4(L/R)^4}{[N^2(L/R)^2+(2M\pi)^2]^2}\right\} = 0. \tag{3.11}$$

A second equation is obtained identical to equation (3.11) with a_{MN} and b_{MN} interchanged which we shall call equation (3.11b). Finally a third equation is obtained which is

$$a_{M0}(M\pi)^4+\frac{d^2a_{M0}}{d\tau^2}+a_{M0}\Gamma^2 4\eta-\left(\frac{\bar{N}_yL^2}{D}\right)\Gamma^2\frac{2[1-(-1)^M]}{M\pi}+\left(\frac{\bar{N}_xL^2}{D}\right)(M\pi)^2a_{M0}$$
$$-4\eta(M\pi)^4N^2(L/R)^2\Gamma\frac{[a_{MN}^2+b_{MN}^2]}{[(M\pi)^2+(L/R)^2N^2]^2}\frac{2}{3}\frac{[1-(-1)^M]}{M\pi}$$
$$+4\eta(M\pi)^4a_{M0}[a_{MN}^2+b_{MN}^2]\left\{\frac{1}{4}+\frac{\frac{1}{8}N^4(L/R)^4}{[N^2(L/R)^2+(2M\pi)^2]^2}\right\}=0. \tag{3.12}$$

In the above equations of motion,

$$\frac{\bar{N}_xL^2}{D}=\left\{[a_{MN}^2+b_{MN}^2]\left[\left(\frac{(M\pi)^2}{8}\right)+\nu N^2\left(\frac{L}{R}\right)^2\frac{1}{8}\right]\right.$$
$$\left.-\nu a_{M0}\frac{[1-(-1)^M]}{M\pi}\Gamma+2a_{M0}^2\frac{(M\pi)^2}{8}\right\}\frac{4\eta}{(1-\nu^2)}, \tag{3.13}$$

$$\frac{\bar{N}_yL^2}{D}=\left\{[a_{MN}^2+b_{MN}^2]\left[N^2\left(\frac{L}{R}\right)^2\frac{1}{8}+\nu\frac{(M\pi)^2}{8}\right]\right.$$
$$+a_{M0}\frac{[1-(-1)^M]}{M\pi}(1-\nu^2)\Gamma-a_{M0}\frac{[1-(-1)^M]}{M\pi}\Gamma$$
$$\left.+\nu^2a_{M0}^2(M\pi)^2\frac{1}{8}\right\}\frac{4\eta}{(1-\nu^2)}. \tag{3.14}$$

From equations (3.11)–(3.14), we see that $a_{MN}\equiv A_{MN/h}$, $b_{MN}\equiv B_{MN/h}$, $a_{M0}\equiv A_{M0/h}$ are functions of time, $\tau\equiv[D/\rho_m hL^4]^{\frac{1}{2}}t$, and the following nondimensional parameters,

$$M,\ N,\ \nu,\ \Gamma,\ L/R.$$

3.2 *Limiting cases*

Now let us consider the limits as $L/R\to 0$ and ∞.

For $L/R\to 0$, equation (3.11) becomes

$$(M\pi)^4a_{MN}+\frac{d^2a_{MN}}{d\tau^2}+\{a_{MN}^2+b_{MN}^2+2a_{M0}^2\}a_{MN}\frac{(M\pi)^4}{8}\frac{4\eta}{(1-\nu^2)}$$
$$+4\eta a_{MN}\frac{[a_{MN}^2+b_{MN}^2]}{16}(M\pi)^4+4\eta a_{M0}^2a_{MN}\frac{(M\pi)^4}{2}=0. \tag{3.15}$$

A similar reduction holds for equation (3.11b). Finally equation (3.12) becomes

$$(M\pi)^4a_{M0}+\frac{d^2a_{M0}}{d\tau^2}+\{a_{MN}^2+b_{MN}^2+2a_{M0}^2\}\frac{(M\pi)^4}{8}a_{M0}\frac{4\eta}{(1-\nu^2)}+(M\pi)^4a_{M0}[a_{MN}^2+b_{MN}^2]=0. \tag{3.16}$$

Note that in this limit each mode can be excited independently, i.e., three possible solutions are

(i) $a_{MN} \neq 0, \quad b_{MN} = a_{M0} = 0,$
(ii) $a_{MN} = 0, \quad b_{MN} \neq 0, \quad a_{M0} = 0,$
(iii) $a_{MN} = b_{MN} = 0, \quad a_{M0} \neq 0,$

where these equations further reduce to

$$\begin{Bmatrix} a_{MN} \\ b_{MN} \end{Bmatrix} (M\pi)^4 + \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\begin{Bmatrix} a_{MN} \\ b_{MN} \end{Bmatrix} + 3(M\pi)^4\left[\frac{3}{4} - \frac{v^2}{4}\right]\begin{Bmatrix} a_{MN}^3 \\ b_{MN}^3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{3.17}$$

$$a_{M0}(M\pi)^4 + \frac{\mathrm{d}^2 a_{M0}}{\mathrm{d}\tau^2} + 3(M\pi)^4 a_{M0} = 0. \tag{3.18}$$

Equation (3.17) is identical with equation (2.6)* for $a/b \equiv 0$, i.e., the two-dimensional limit of our flat plate solution. Equation (3.18) is the result for a two-dimensional plate with no spanwise bending and zero spanwise strain, cf., coefficients listed in Table 1. It is clear that the equations of motion derived here correctly reduce to those of a two-dimensional plate.

For $L/R \to \infty$, we multiply equation (3.11) and (3.12) by R^4/L^4N^4 and then let $R/L \to 0$. The results are

$$a_{MN} + \frac{\mathrm{d}^2 a_{MN}}{\mathrm{d}\tau_N^2} + \left\{\frac{[a_{MN}^2 + b_{MN}^2]}{8} + a_{M0}\frac{[1-(-1)^M]}{M\pi}(1-v^2)\varepsilon^{-\frac{1}{2}} - a_{M0}\frac{[1-(-1)^M]}{M\pi}\varepsilon^{-\frac{1}{2}}\right\}$$
$$\times \frac{4\eta}{(1-v^2)}a_{MN} + 4\eta a_{MN}\frac{[a_{MN}^2 + b_{MN}^2]}{16} - 4\eta(\tfrac{4}{3})\frac{[1-(-1)^M]}{M\pi}\varepsilon^{-\frac{1}{2}}a_{M0}a_{MN} = 0, \tag{3.19}$$

$$\frac{\mathrm{d}^2 a_{M0}}{\mathrm{d}\tau_N^2} + a_{M0}\frac{4\eta}{\varepsilon} - \left\{\frac{[a_{MN}^2 + b_{MN}^2]}{8} + a_{M0}\frac{[1-(-1)^M]}{M\pi}(1-v^2)\varepsilon^{-\frac{1}{2}} - a_{M0}\frac{[1-(-1)^M]}{M\pi}\varepsilon^{-\frac{1}{2}}\right\}$$
$$\times \frac{4\eta}{(1-v^2)}\varepsilon^{-\frac{1}{2}}2\frac{[1-(-1)^M]}{M\pi} = 0. \tag{3.20}$$

Note that if M is even then there is no coupling between a_{M0} and a_{MN} or b_{MN}. This leads to a hardening nonlinearity of the type previously found for a flat plate when M is even. By contrast Evensen's solution using a different modal expansion for w indicates a softening nonlinearity for all M, see Section 4, equation (4.1). A possible explanation for this difference is the retention of only a single axial mode in the present analysis. If additional axial modes were retained (which is what Evensen has effectively done) then it is possible the present result would be modified. In particular it should be noted that when only one axial mode is retained, for M odd the Nth and zeroth circumferential modes are coupled while they are uncoupled for M even. This indicates the desirability of retaining additional axial (M) modes for M even in order to allow for the effect of circumferential coupling.

Let us now consider $M = 1$ which we may expect will correspond most closely to a ring. For $M = 1$, equations (3.19) and (3.20) become

$$a_{1N} + \frac{\mathrm{d}^2 a_{1N}}{\mathrm{d}\tau_N^2} + [a_{1N}^2 + b_{1N}^2]a_{1N}\eta\left\{\frac{1}{2(1-v^2)} + \frac{1}{4}\right\} - a_{1N}a_{10}\varepsilon^{-\frac{1}{2}}4\eta\left\{\frac{8}{3\pi} + \frac{v^2}{(1-v^2)}\frac{2}{\pi}\right\} = 0 \tag{3.21}$$

$$\frac{\mathrm{d}^2 a_{10}}{\mathrm{d}\tau_N^2} + a_{10}4\eta\varepsilon^{-1}\left\{1 + \frac{8}{\pi^2}\frac{v^2}{(1-v^2)}\right\} - [a_{1N}^2 + b_{1N}^2]\eta\varepsilon^{-\frac{1}{2}}\left\{\frac{2}{\pi(1-v^2)}\right\} = 0. \tag{3.22}$$

* For $M = 1$, of course.

Recalling the equations for a ring, equation (2.6), we see that equation (3.21) and (3.22) are of the same form; however, while the { } quantities are exactly one for the ring, they are here for $\nu = 0{\cdot}3$,

$$\left\{\frac{1}{2(1-\nu^2)}+\frac{1}{4}\right\} = 0{\cdot}8,$$

$$\left\{\frac{8}{3\pi}+\frac{\nu^2}{(1-\nu^2)}\frac{2}{\pi}\right\} = 0{\cdot}91,$$

$$\left\{1+\frac{8}{\pi^2}\frac{\nu^2}{(1-\nu^2)}\right\} = 1{\cdot}08,$$

$$\left\{\frac{2}{\pi(1-\nu^2)}\right\} = 0{\cdot}7.$$

These differences may be explained by considering a similar situation for the flat plate. There we saw that, as we take the two-dimensional limit of the three-dimensional solution, the coefficient of the nonlinear term was somewhat different from that deduced for a two-dimensional plate with no spanwise bending, $\partial w/\partial y = 0$. Even for the case of two-dimensional bending, depending on whether one assumes zero spanwise stress or zero strain, the coefficient differs. A similar situation apparently exists for the ring limit. The ring solution is for no axial bending (and sero axial stress) and hence does not exactly correspond to the $L/R \to \infty$ limit of the cylindrical shell solution.

Recently Evensen (personal communication) has also considered this limit by simplifying the initial equations, (3.1) and (3.2).

4. COMPARISON WITH SOLUTIONS BY OTHER AUTHORS

Evensen [1, 2] and Dowell [9] have discussed the inadequacies of previous analyses by Chu [12] and Nowinski [13] elsewhere. Hence they will not be discussed here. Evensen [2] has recently presented an improved analysis of the problem which we shall compare to the present one. First we shall outline his analysis. It begins with equation (3.1) and (3.2). An expansion is assumed as follows.

$$w(x,y,t) = \left[A_{MN}\cos\frac{Ny}{R}+B_{MN}\sin\frac{Ny}{R}\right]\sin\frac{M\pi x}{L}+\frac{N^2}{4R}[A_{MN}^2+B_{MN}^2]\sin^2\frac{M\pi x}{L}. \quad (4.1)$$

This was selected by analogy to the result obtained by Evensen for the ring, equation (2.9). Substituting equation (4.1) into equation (3.2), Φ may be determined. Only the particular part of the solution is used. It is assumed that $\Phi_{\text{homogeneous}} = 0$. Knowing Φ and w, it is then determined a posteriori that v is continuous in the circumferential variable, i.e.,

$$\int_0^{2\pi R}\frac{\partial v}{\partial y}\mathrm{d}y = \int_0^{2\pi R}\left[\frac{(Ny-\nu Nx)}{Eh}+\frac{w}{R}-\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\mathrm{d}y = 0. \quad (4.2)$$

This verifies the selection of the modal expansion in as far as this continuity is concerned.

Unfortunately, as Evensen himself points out, the boundary condition of zero moment at $x = 0, L$

$$\frac{\partial^2 w}{\partial x^2} = 0$$

is not satisfied by equation (4.1). Also, as Evensen pointed out, there is no mechanism for satisfying axial in-plane constraints since the homogeneous stress function solution has been neglected. The equations derived in Section 3 are free from both of these defects.

Let us now complete the outline of Evensen's derivation and compare the final equations of motion with equations (3.11)–(3.12). Substituting equation (4.1) and the derived solution for Φ_p into equation (3.1), Evensen uses Galerkin's method to obtain,

$$\frac{d^2 a_{MN}}{d\tau_{MN}^2} + a_{MN} + \frac{3}{8}\varepsilon a_{MN}\left[a_{MN}\frac{d^2 a_{MN}}{d\tau_{MN}^2} + \left(\frac{da_{MN}}{d\tau_{MN}}\right)^2 + b_{MN}\frac{d^2 b_{MN}}{d\tau_{MN}^2} + \left(\frac{db_{MN}}{d\tau_{MN}}\right)^2\right]$$
$$-\varepsilon\gamma a_{MN}[a_{MN}^2 + b_{MN}^2] + \varepsilon\delta b_{MN}[a_{MN}^2 + b_{MN}^2] = 0 \quad (4.3)$$

and another equation identical to equation (4.3) with a_{MN} and b_{MN} interchanged. In the above

$$\tau_{MN} \equiv \omega_{MN} t,$$

$$\omega_{MN}^2 \equiv \frac{E}{\rho_m R^2}\left[\frac{\xi^2}{(\xi^2+1)^2} + \varepsilon\frac{(\xi^2+1)^2}{12(1-\nu^2)}\right],$$

$$\xi \equiv \frac{M\pi/L}{N/R},$$

$$\varepsilon \equiv \left(\frac{N^2 h}{R}\right)^2,$$

$$\gamma \equiv \frac{\xi^4\left[\frac{1}{(\xi^2+1)^2} - \frac{1}{16} - \frac{\varepsilon}{12(1-\nu^2)}\right]}{\left[\frac{\xi^4}{(\xi^2+1)^2} + \varepsilon\frac{(\xi^2+1)^2}{12(1-\nu^2)}\right]},$$

$$\delta \equiv \frac{3\xi^4}{16}\frac{\left[\frac{1}{(\xi^2+1)^2} + \frac{1}{(9\xi^2+1)^2}\right]}{\left[\frac{\xi^4}{(\xi^2+1)^2} + \varepsilon\frac{(\xi^2+1)^2}{12(1-\nu^2)}\right]}.$$

In this formulation, of course, there is no equation of motion for a_{M0}.

Let us now compare Evensen's result with the one derived in Section 3. It is difficult to compare (4.3) with (3.11)–(3.12) in general (except numerically, of course). However, the limiting cases as $L/R \to 0$ and $L/R \to \infty$ can be examined rather readily.

Evensen has shown that as $L/R \to \infty$, $\xi \to 0$ and $\gamma \to 0$, $\delta \to 0$. Hence equation (4.3) becomes

$$\frac{d^2 a_{MN}}{d\tau_{MN}^2} + a^{MN} + \frac{3}{8}\varepsilon a_{MN}\left[a_{MN}\frac{d^2 a_{MN}}{d\tau_{MN}^2} + \left(\frac{da_{MN}}{d\tau_{MN}}\right)^2 + b_{MN}\frac{d^2 b_{MN}}{d\tau_{MN}^2} + \left(\frac{db_{MN}}{d\tau_{MN}}\right)^2\right] = 0, \quad (4.4)$$

where

$$\omega_{MN}^2 \to \frac{E}{\rho_m R^2} \frac{\varepsilon}{12(1-\nu^2)} = \frac{DN^4}{\rho_m h R^4} = \omega_N^2.$$

Equation (4.4) is of the same form as the equations previously obtained by Evensen [8] for a ring except that for the ring the constant $\frac{3}{8}$ was instead $\frac{1}{2}$. Hence, Evensen's equations for the cylindrical shell reduce approximately to his ring equations. It is not thought that the ring equations should be the precise limit for $L/R \to \infty$, though one would expect the full equations to reduce to a similar form as indeed they appear to do for both Evensen's equations and those of this present analysis. See [9] for a discussion of the equivalency of the two different approaches for the ring equations.

Now consider the other limit for Evensen's equations, $L/R \to 0$. As $L/R \to 0$, $\xi \to \infty$, $\delta \to 0$, $\varepsilon \to 0$, and

$$\gamma \to -\frac{1}{16} \frac{12(1-\nu^2)}{\varepsilon}$$

Equation (3.3) becomes

$$\frac{d^2 a_{MN}}{d\tau^2} + a_{MN} + \tfrac{3}{4}(1-\nu^2) a_{MN}[a_{MN}^2 + b_{MN}^2] = 0. \tag{4.5}$$

This agrees with the result obtained from the present analysis if $\bar{N}_x = 0$. For $\bar{N}_x \neq 0$, the coefficient of the nonlinear term from the present analysis [cf., equation (3.17)] is

$$\tfrac{3}{4}(3-\nu^2)$$

This compares with the limit of Evensen's equation [cf. (4.5)]

$$\tfrac{3}{4}(1-\nu^2).$$

Hence, there is roughly a factor of three difference in the coefficients. Since the former coefficient also may be derived from an analysis for a two-dimensional plate, it seems clear the discrepancy is due to the omission of in-plane restraint in the analysis of [2].

Thus it is concluded that the present analysis is equally accurate to that of Evensen for $L/R \to \infty$ and is more accurate for $L/R \to 0$.

Finally we mention a recent report by Mayers and Wrenn [14] which also describes an analysis which is intended to improve upon Evensen's solution. The principal features of their investigation are:

(1) A study of the energy of the oscillation is made, particularly with respect to finding the mode of minimum energy. Presumably such considerations might be important to determining the response of the system to "white noise" (equal energy per frequency band) inputs.

(2) They use a somewhat more general assumed deflection form than Evensen; however, axial boundary conditions are not satisfied. (No homogeneous stress function solution is determined or used.) In items (1) and (2), the Donnell equations are employed.

(3) A more accurate set of shell equations (Sander's theory) is also considered. The equations are expressed in terms of the three displacements of the shell. The latter are expanded in a Fourier Series, however, only one term is retained for the radial displacement.

The expansion can be expressed in terms of the present notation as follows

$$w = A_{MN} \cos \frac{Ny}{R} \sin \frac{M\pi x}{L}.$$

Since it is known that this expansion is inadequate in some circumstances, the gain in accuracy associated with a more accurate shell theory has been offset by a less complete expression for the radial displacement. The study of [14] does suggest that the investigation of more accurate shell theories beyond that of Donnell may be of considerable interest.

5. CONCLUDING REMARKS

The principal conclusions to be drawn from the present study are

(1) the method of "averaged in-plane boundary conditions" generally yields good results, and

(2) the modal equations derived are accurate in the limits of $L/R \to \infty$ and $L/R \to 0$, unlike previously available results.

The principal limitation of the present analysis would appear to be the use of the Donnell shell equations rather than a more accurate set such as those due to Sanders [14].

Finally although it is not our purpose here to obtain numerical results from the equations of motion, we will briefly discuss one aspect of the equations which may influence the solution technique employed.

If one is investigating the forced response near resonance or the natural frequency-amplitude relation, the harmonic balance or so-called method of averaging may be used as given in any number of textbooks. See, e.g. [15]. Evensen [2, 8, 11] and Dowell [9] have used this technique. Olson and Fung [15] have also used this technique along with Evensen's approach to the equations of motion to study the flutter of a cylindrical shell using two axial modes. If, as in the flutter problem, one wishes to retain several axial modes in the w expansion, the harmonic balance technique becomes very tedious and one may wish to consider numerical integration of the equations of motion to obtain a time history of the displacement. In this regard, it should be pointed out that since the natural frequency of the A_{M0} mode in the present analysis will be considerably higher than that of the A_{MN} or B_{MN} modes, a somewhat smaller time step will be needed for the integration than that indicated by the flutter frequency.

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Абстракт—Дается обзор модальных приближений вывода уравнений движения для нелинейных изгибных колебаний цилиндрической оболочки. Представляется и используются улучшеный способ, который выполняет точнее граничные услоия задачи. Приводятся сравнения приведенных уравнений с выведенными ранее в литературе.